Introduction to Order Book Dynamics

High Frequency Trading in Foreign Exchange

Tom J. Espel

Hong Kong University of Science and Technology

Tuesday 7th March, 2023

Outline

Foreign Exchange Market

Do you know how large the FX market is in volume?

Foreign Exchange Market

It is the world's largest market, with approximately
6,600,000,000,000 USD
traded per day (BIS 2016 estimate).

Outline

Foreign Exchange Market Overview

Some of the distinctive characteristic of the FX market are the following.

- It operates **24 hours a day, 5 days a week**.
- It is **decentralised**, which means that there can be several competing prices for a given asset.
- Its **main hub is London**, other major hubs are New York and Hong Kong. Dubai and Singapore are other important centres.
- The most traded currencies are the US Dollar (USD), the Euro (EUR), the Japanese Yen (JPY) and the British Pound Sterling (GBP).

Foreign Exchange Market Structure

- A small groups of banks make up the interbank foreign exchange market.
- Structurally, it is therefore a **brokerage** market. It means that the end users will not directly transact with each other.
- End users comprise companies, governments, pension funds, etc. The retail market is very small in comparison to the wholesale market.
- FX is a key component of **macro trading**.

Foreign Exchange Market Structure

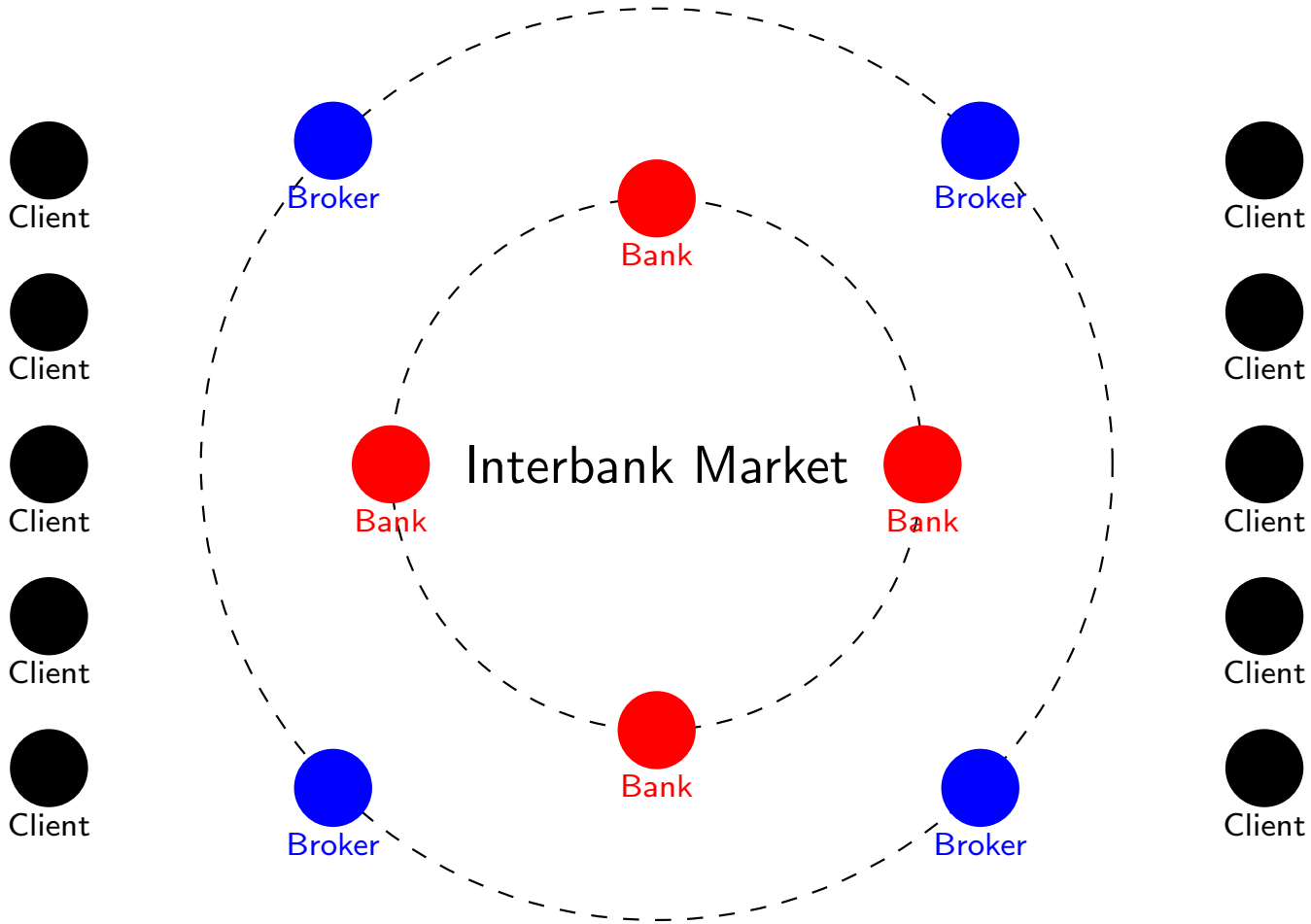

Figure 1: FX Market Structure

Foreign Exchange Market Structure

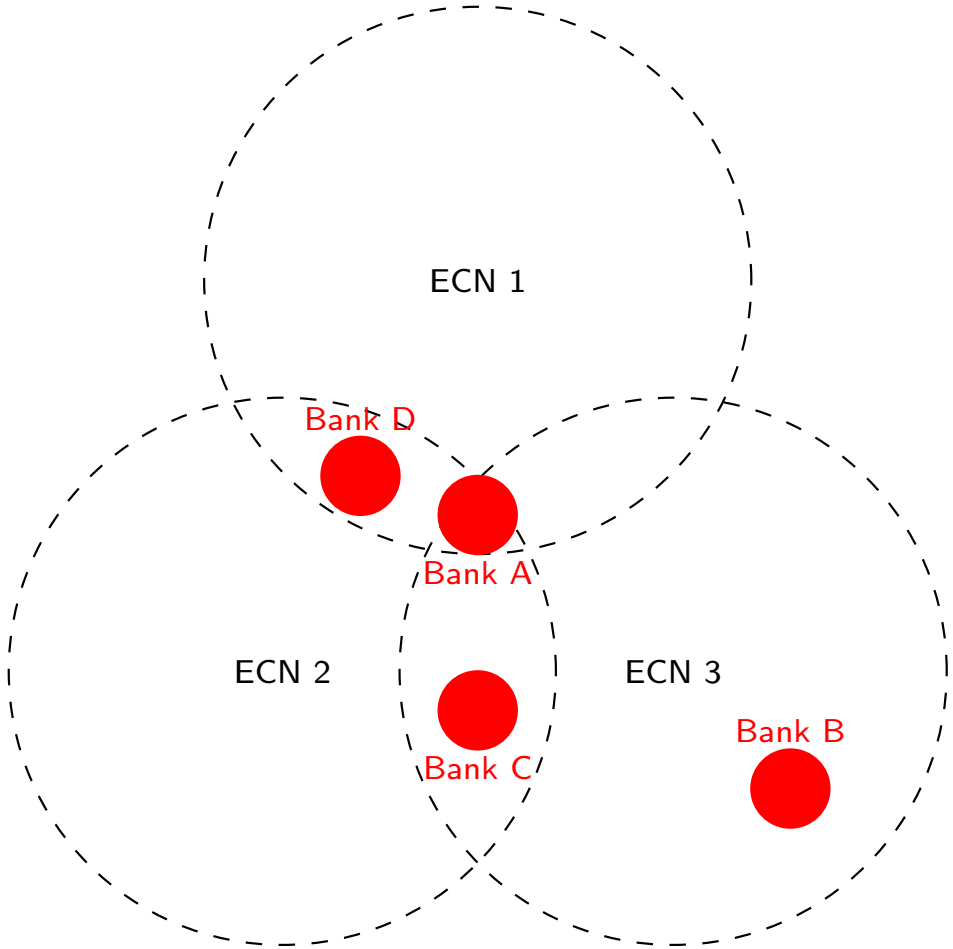

Figure 2: eFX Interbank Market Structure - Fragmentation

High Frequency Trading

- Order of magnitude below a few seconds. Most often down to ms.
- Grew in the mid 00s, became most common in the 10s.
- Highly automated asset classes: equities and foreign exchange.

High Frequency Trading

Not all assets can be automatically traded. The requirements are as follows.

1. An electronic marketplace for one or several **standardised** versions of the product.
2. An infrastructure that allows market participants to receive and send information reliably.
3. Other counterparties willing to trade electronically.

Electronic trading can become high frequency trading based on the connection (throttling) and algorithmic speeds involved (simple or actively quoted products).

High Frequency Trading

- There is a significant cost of entry for electronic market participants.
- Cost is even higher for high frequency (specialised software and hardware).
- Only financially viable for very active markets, with tight margins and large volumes.

High Frequency Trading

Implications of high frequency trading.

- Overall more efficient markets, in particular on optimal execution.
- At microstructure level: new properties appear. Active field of research.
- Algo versus algo interactions, which can be complex to understand.
  Flash crash on S&P500 of May 6th, 2010.

Outline

What is an Order Book?

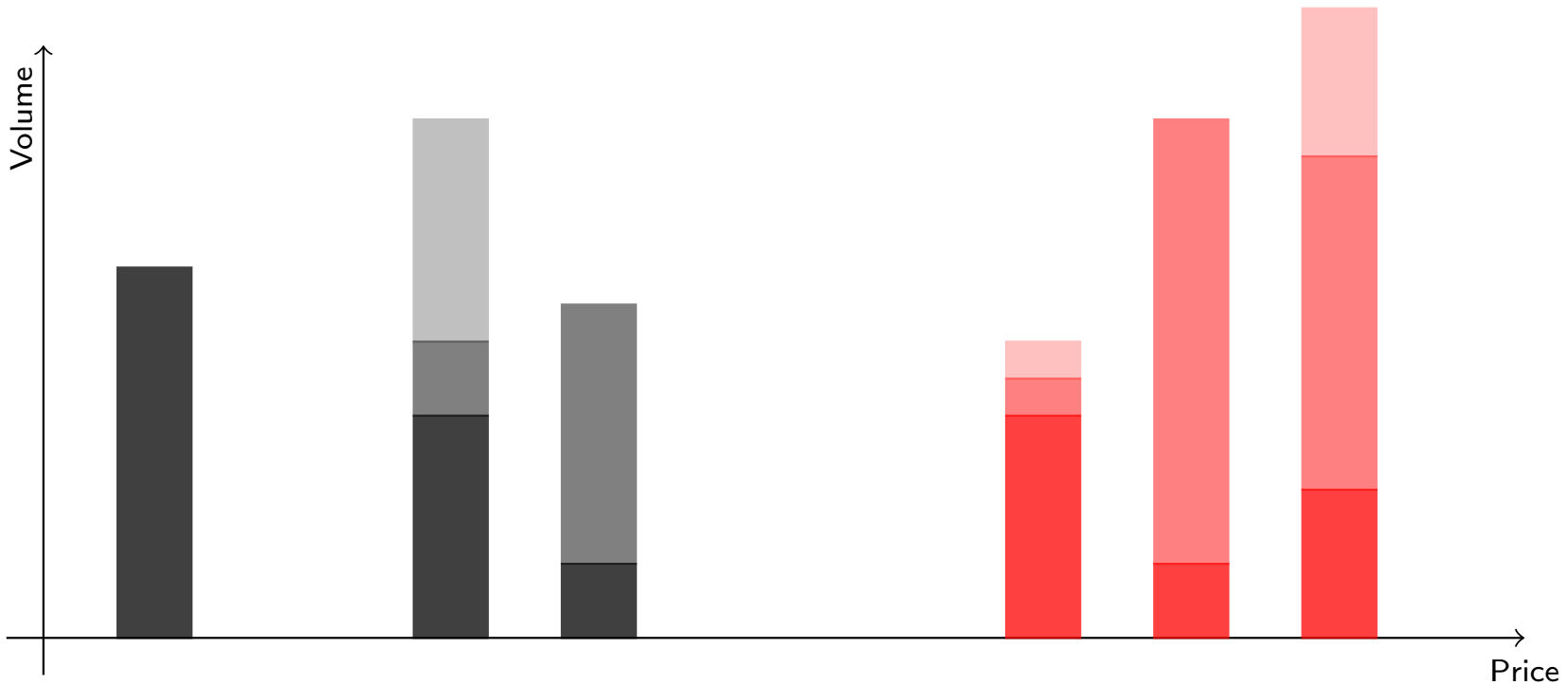

Figure 3: Typical Representation of a Limit Order Book

What is an Order Book?

Definition (Limit Order Book)

The Limit Order Book (LOB) is the collection of the open interests, for a given asset, at a given point in time.

- There can be multiple order books maintained in parallel by different venues.
- Not all interests are visible to all market participants in the orderbook.
- By definition, the limit orders on the limit order book have not been executed.

What is an Order Book?

- There are different possible execution priorities, but the most common is price / time.
- This means that orders are executed sequentially, starting with the ones with the best price, and when several have the same price, starting by the oldest orders.
- The system that executes the orders is called the **matching engine**.

What is an Order Book?

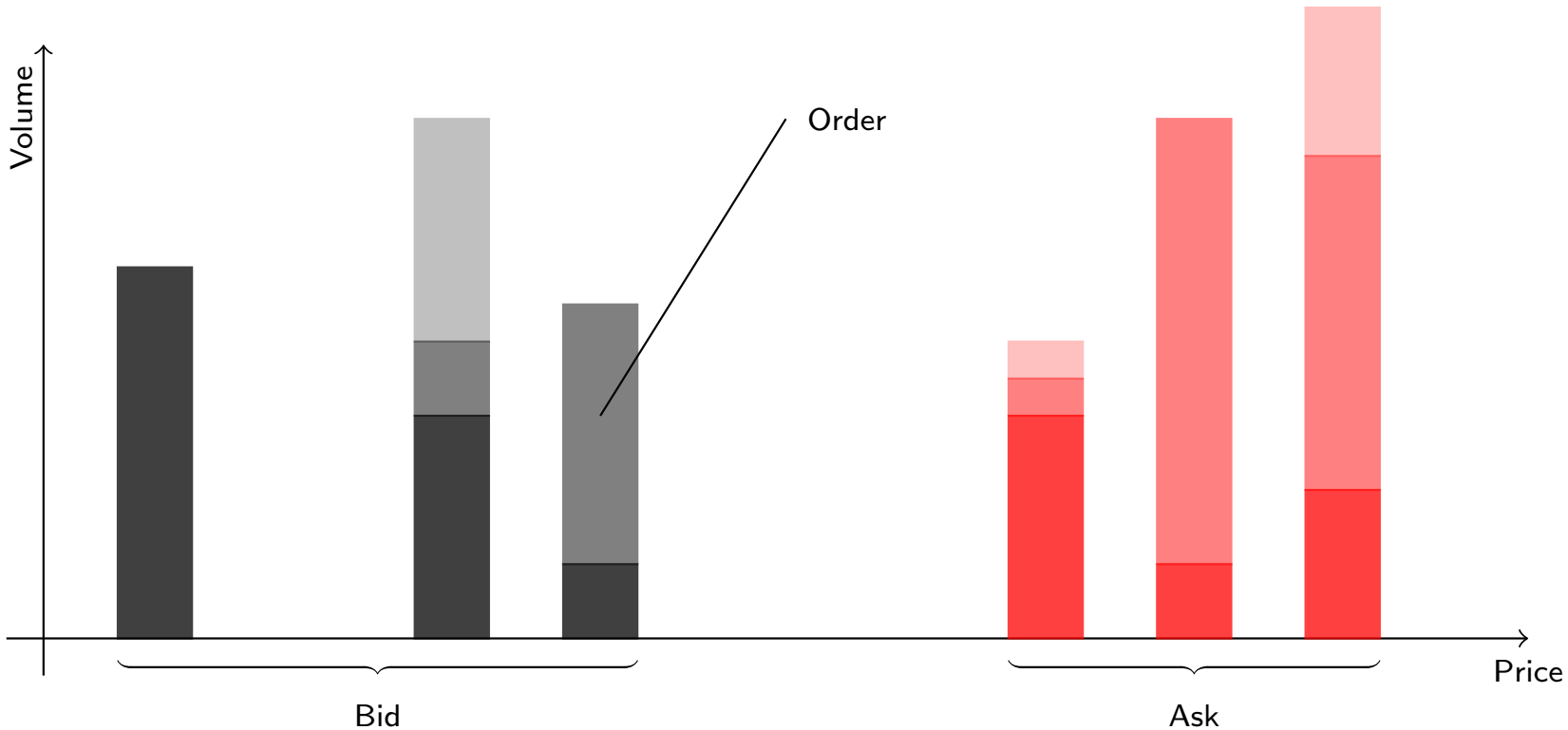

Figure 4: Typical Representation of a Limit Order Book

What is an Order Book?

We can define the following concepts[1].

- Open interest to buy the asset are called **bids**. We write the corresponding prices at time t as the price-decreasing ordered sequence $(b_i(t))_{i\in\mathbb{N}}$.
- Open interest to sell are called **asks** or **offers**. We write the corresponding prices at time t as the price-increasing ordered sequence $(a_i(t))_{i\in\mathbb{N}}$.

What is an Order Book?

Definition (Order Book Spread)

The difference between the best ask price and the best bid price is called the spread.

$$s(t) = a_0(t) - b_0(t)$$

Definition (Order Book Mid)

The mid price is the average of the best bid price and the best ask price.

$$m(t) = \frac{a_0(t) + b_0(t)}{2}$$

What is an Order Book?

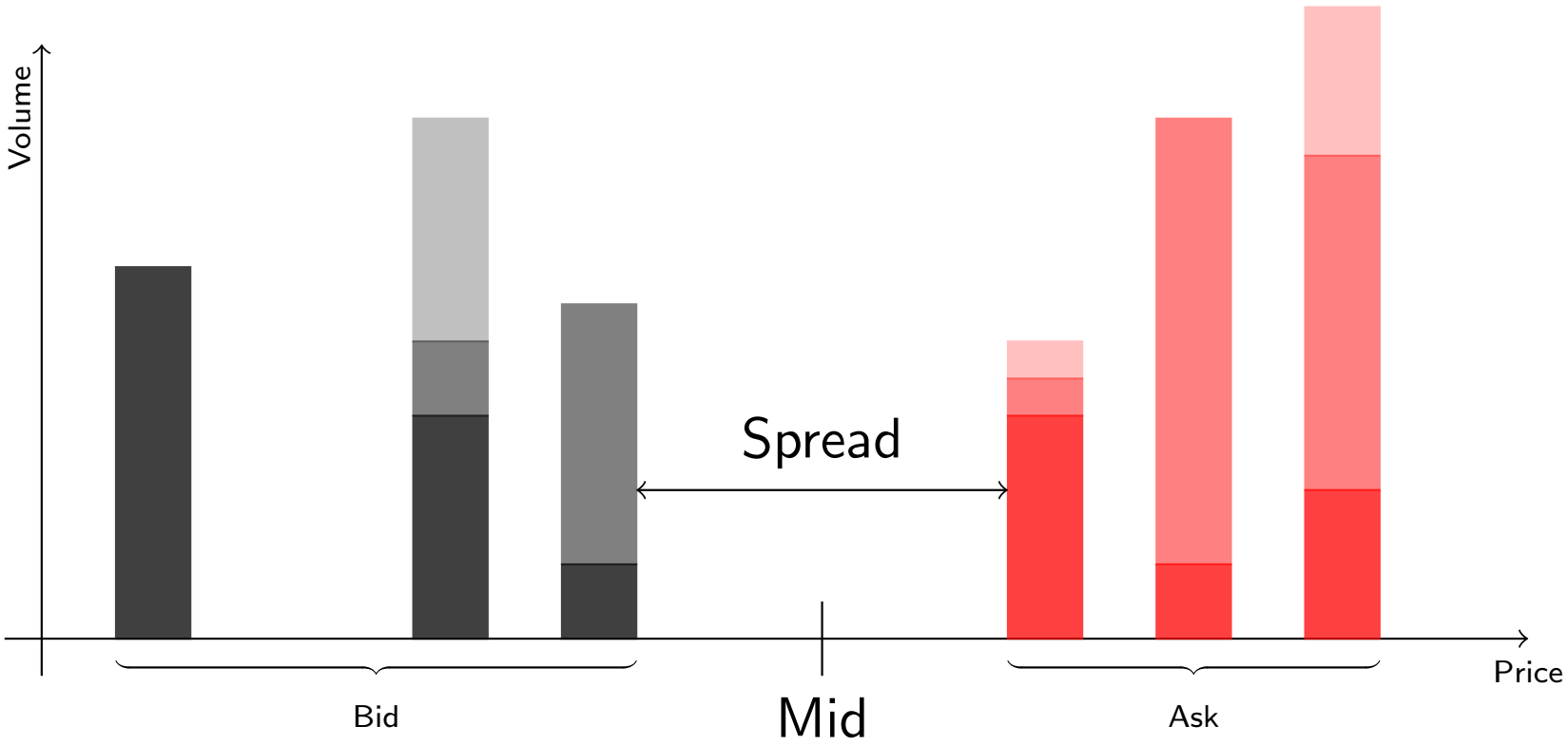

Figure 4: Typical Representation of a Limit Order Book

Elements of Market Microstructure

Definition (Return)

$$r(t) = \frac{m(t)}{m(t-1)}$$

A more compact and common representation is the log return.

$$\begin{aligned} \log r(t) &= \log \frac{m(t)}{m(t-1)} \\ &= \log\left(1 + \varepsilon_m(t)\right) \\ &\overset{\text{Taylor}}{\simeq} \varepsilon_m(t) \end{aligned}$$

Market Microstructure: Stylised Facts

Definition (Stylised Facts)

Stylised facts are statistical properties observed on asset returns.

There are several commonly accepted stylised facts [2] but two of interest today are.

1. **Aggregational Gaussianity** as the returns' timescale increases, the distribution looks more like a Gaussian distribution
2. **Heavy tails** the distribution of the returns has power-law tails, in particular for smaller returns timescales

Market Microstructure: Random Walk

- The dynamics of the mid price $(m(t_n))_{n\in\mathbb{N}}$ are often represented as a random walk.
- Random walks are processes with random **steps**.
- The most commonly used is the **Gaussian Random Walk**.

Market Microstructure: Random Walk

Definition (Martingale)

Let $(X_i)_{i\in\mathbb{N}}$ a sequence of random variables.

$$\mathbf{E}|X_n| < \infty$$
$$\mathbf{E}(X_{n+1}|X_{i\leq n}) = X_n$$

$(X_i)_{i\in\mathbb{N}}$ is a martingale.

Martingale are discrete-time processes[3], but there is also a continuous-time generalisation.

Market Microstructure: Random Walk

Definition (Gaussian Random Walk)

Let $(X_i)_{i\in\mathbb{N}}$ independent identically distributed variables such that $X_i \hookrightarrow \mathcal{N}(0,1)$. We call Gaussian Random Walk the process $(W_i)_{i\in\mathbb{N}}$ such that.

$$W_0 = 0$$
$$W_k = W_{k-1} + X_k, \forall k \in \mathbb{N}^*$$

Trivially, $(W_i)_{i\in\mathbb{N}}$ is a martingale.

Market Microstructure: Random Walk

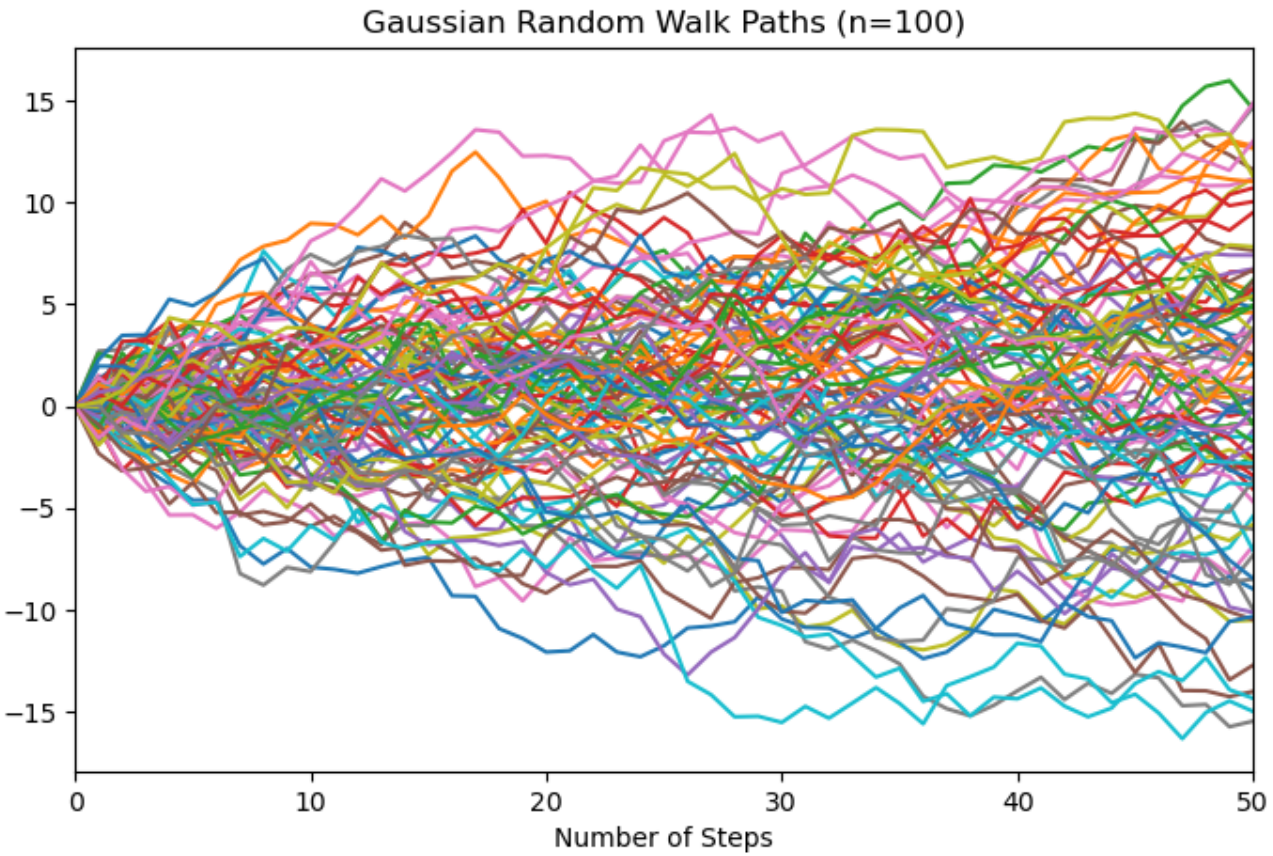

Figure 5: Example of Gaussian Random Walk Simulation

Market Microstructure: Random Walk

- The price formation is a real, continuous process.
- The market information is discrete, because it is downsampled.
- Financial mathematics for medium to high frequency will focus on continuous processes.

Market Microstructure: Random Walk

An example of continuous martingale is the Wiener Process.

Definition (Wiener Process)

The Wiener Process $W(t)$ is a stochastic process such that.

1. **Initialisation** $W(0) = 0$
2. **Independent Increments** $\forall t > 0, \delta \geq 0, s < t$ the increments $W(t+\delta) - W(t)$ are independent of $W(s)$
3. **Gaussian Increments** $\forall t > 0, \delta \geq 0$, $(W(t+\delta) - W(t)) \hookrightarrow \mathcal{N}(0, \delta)$
4. **Continuity** $W(.) \in \mathcal{C}^0(\mathbb{R})$

High Frequency Order Book: Downsampling

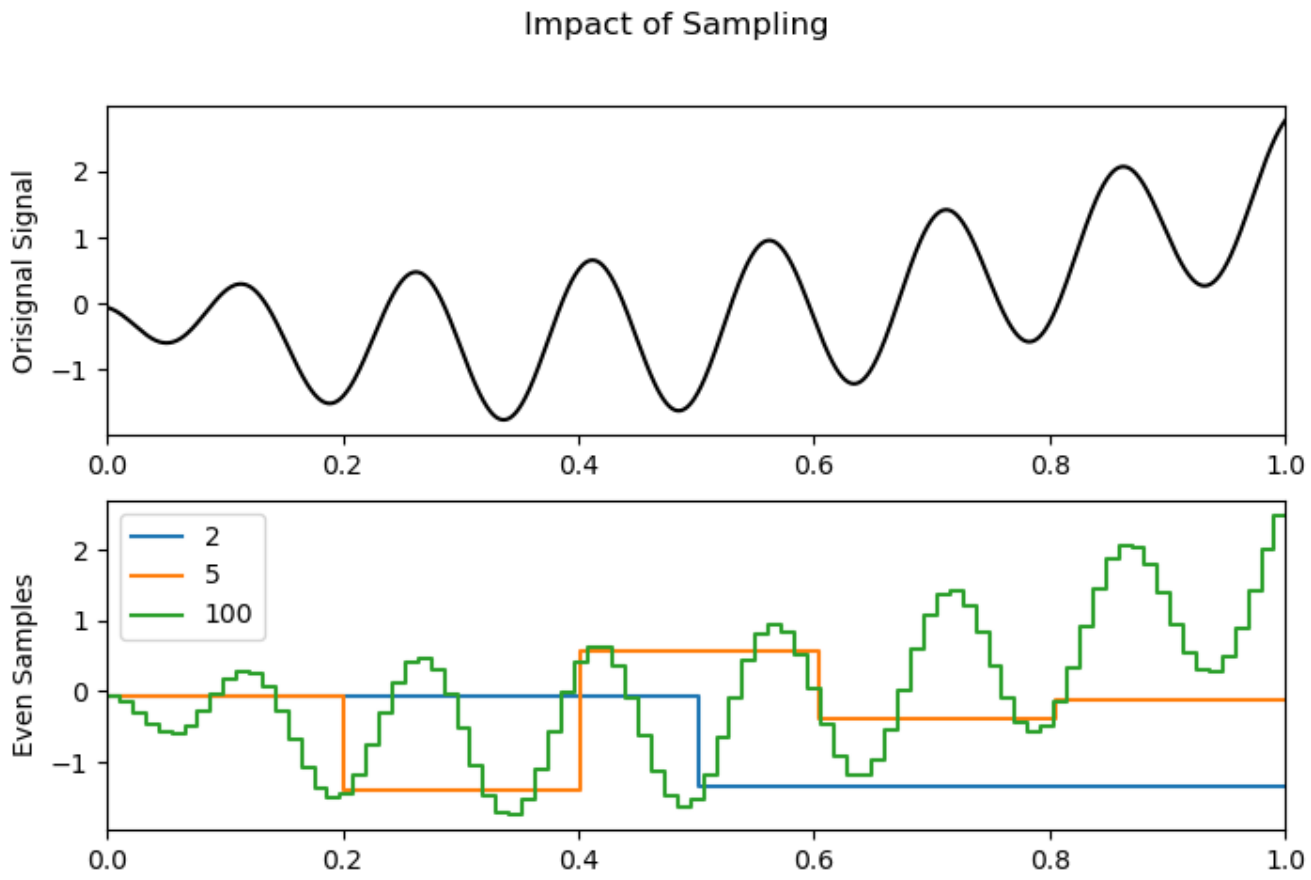

Figure 6: Examples of Sampling

High Frequency Order Book: Downsampling

- We've seen that **continuous** processes are used to represent financial information.
- At low frequency, we can find a downsampling pace such that there is a regular number of updates. The only boundary for information theory is the Nyquist-Shannon theorem[4].

Theorem (Nyquist-Shannon)

If a signal s contains no higher frequencies than $f > 0$, it can be downsampled at a frequency $f_s \geq 2f$.

- It is not the case at high frequency.

High Frequency Order Book: Downsampling

The mathematical tools used to link the continuous process and the samples in high frequency are called the point processes. The most famous one is the Poisson process.

Definition (Poisson Distribution)

$X \hookrightarrow \mathcal{P}(\lambda)$ with $\lambda > 0$ its rate and

$$\mathbf{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

High Frequency Order Book: Downsampling

Definition (Poisson Point Process)

The counting process $N(t), t \leq 0$ is a Poisson process with rate $\lambda > 0$ when it satisfies.

1. $N(0) = 0$
2. The increments are independent
3. $\forall t \geq s \geq 0, N(t-s) \hookrightarrow \mathcal{P}(t-s)$

High Frequency Order Book: Fat Tails

The Student t-distribution is used to represent fat tails.

Definition (Student t-Distribution)

X follows a Student t-distribution with degree of freedom $\nu \in \mathbb{R}_+^*$ when its probability density function is

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

with Γ the gamma function.

An interesting property of the Student t-distribution is that it converges to a Normal distribution for very large degrees of freedom ν.

High Frequency Order Book: Fat Tails

- **Aggregational Gaussianity** as the returns' timescale increases, the distribution looks more like a Gaussian distribution[2].
- An interesting property of the Student t-distribution is that it converges to a Normal distribution for very large degrees of freedom ν.

Conclusion

- FX is a major market with some assets being traded at high frequency.
- The **order book** is one of the main sources of information and displays both high and low frequency properties.
- Quantitative finance uses those results to **explain, quantify and predict** market microstructure behaviours.
- Today we've explored some of those results for the returns on the mid price.

Bibliography


O. Guéant, *The Financial Mathematics of Market Liquidity: From optimal execution to market making*, vol. 33.
CRC Press, 2016.


R. Cont, "Empirical properties of asset returns: stylized facts and statistical issues," *Quantitative finance*, vol. 1, no. 2, p. 223, 2001.


S. Shreve, *Stochastic calculus for finance I: the binomial asset pricing model*.
Springer Science & Business Media, 2005.


C.-A. Lehalle and S. Laruelle, *Market microstructure in practice*.
World Scientific, 2018.

Thank you! :-)